OLIMPIADA NAŢIONALĂ DE MATEMATICĂ

Etapa locală-16.02.2019

Clasa a IX-a

Barem

1. i) $\left[\frac{2x+1}{3}\right] = \frac{3x+1}{2} = k$, $k \in \mathbf{Z} \Rightarrow$

$k \le \frac{2x+1}{3} < k+1$ 1p

$x = \frac{2k-1}{3}$, $k \in \mathbf{Z}$ 1p

$k \in \{-1, 0\} \Rightarrow x \in \{-1, -\frac{1}{3}\}$...2p

ii) Fie $a_k = \{k\sqrt{2}\}$, $k \in \mathbf{N}^*$ si $k \neq i$

Dacă $a_k = a_i \Rightarrow k\sqrt{2} - [k\sqrt{2}] = i\sqrt{2} - [i\sqrt{2}] \Rightarrow \sqrt{2} = \frac{[k\sqrt{2}] - [i\sqrt{2}]}{k-i} \in \mathbf{Q}$ (fals)

Deci $k \neq i \Rightarrow a_k \neq a_i$ şi $a_k \in (0, 1)$

$a_1, a_2, a_3, \ldots, a_{10^{p+n}}, a_{10^{p+n}+1} \in (0, 1) \Rightarrow$

$(\exists)$ $i, k \in \mathbf{N}^*$, $i \neq k$ astfel încât $|a_i - a_k| \le \frac{1}{10^{p+n}}$

$\left|[k\sqrt{2}] - [i\sqrt{2}] + \sqrt{2}(i-k)\right| \le \frac{1}{10^{p+n}}$ $\quad | \cdot 10^p$

$\left|10^p([k\sqrt{2}] - [i\sqrt{2}]) + 10^p(i-k)\sqrt{2}\right| \le \frac{1}{10^n}$

Alegem $a = 10^p([k\sqrt{2}] - [i\sqrt{2}])$ și

$b = 10^p(i-k)$...3p

2.i) a soluție $\Rightarrow a^2 = a + 1 \Rightarrow a^5 = 5a + 3$...2p

$a^5 + b^5 = 5(a+b) + 6 = 5 \cdot 1 + 6 = 11$ 1p

ii) $P(n) : 8^n + 42n - 50 \vdots 49$

$P(0) : 8^0 + 42\cdot 0 - 50 = -49 \vdots 49$ (adev.) 1p

$P(n) \rightarrow P(n+1)$

$P(n+1) : 8^{n+1} + 42(n+1) - 50 \vdots 49$ 1p

Notăm $x_n = 8^n + 42n - 50$

$x_{n+1} - 8x_n = 8^{n+1} + 42(n+1) - 50 - 8^{n+1} - 8\cdot 42n + 400 = -7\cdot 42n + 392 =$

$= -49\cdot 6n + 49\cdot 8 \Rightarrow x_{n+1} = 8x_n - 49\cdot 6n + 49\cdot 8 \vdots 49 \Rightarrow P(n+1)$ adev 2p

2. Fie $a_m = \sqrt{3}$, $a_n = \sqrt{5}$, $a_p = \sqrt{7}$, $m < n < p$

$\sqrt{3} = a_1 \cdot r^{m-1}$, $\sqrt{5} = a_1 \cdot r^{n-1}$, $\sqrt{7} = a_1 \cdot r^{p-1}$ 2p

$\frac{\sqrt{5}}{\sqrt{3}} = r^{n-m}$, $\frac{\sqrt{7}}{\sqrt{5}} = r^{p-n}$ 1p

$r^{2\,(n-m)\,(p-n)} = \frac{5^{p-n}}{3^{p-n}} = \frac{7^{n-m}}{5^{n-m}}$ 3p

$5^{p-m} = 3^{p-n} \cdot 7^{n-m}$, cu $p-m, p-n, n-m \in \mathbf{N}^*$ $\Rightarrow 5 \vdots 3$ (fals) $\Rightarrow$

Numerele $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ nu pot fi termeni oarecare într-o progresie geometrică...1p

3.i) $\overrightarrow{DR} = 2\,\overrightarrow{RC}$ ……………………………..2p

$\overrightarrow{AC} = \overrightarrow{AR} + \overrightarrow{RC} = \overrightarrow{AR} + \frac{1}{2}\overrightarrow{DR}$ ………………………………………………..2p

ii) $\overrightarrow{AR} = \overrightarrow{AC} - \frac{1}{3}\overrightarrow{AB} = \overrightarrow{AD} + \frac{2}{3}\overrightarrow{AB}$ ……………………………………………...1p

$\frac{TD}{TB} = \frac{2}{3} \Rightarrow \overrightarrow{AT} = \frac{1}{1+\frac{2}{3}}\overrightarrow{AD} + \frac{\frac{2}{3}}{1+\frac{2}{3}}\overrightarrow{AB} = \frac{3}{5}\overrightarrow{AD} + \frac{2}{5}\overrightarrow{AB}$ ……………………….....1p

$\overrightarrow{AT} = \frac{3}{5}\left(\overrightarrow{AD} + \frac{2}{3}\overrightarrow{AB}\right) = \frac{3}{5}\overrightarrow{AR} \Rightarrow$ A, T, R sunt coliniare …………………….….1p